J. Fluid Mech. (1976), *vol.* 77, *part* 1, *pp.* 1–25
Printed in Great Britain

A theory for wave-power absorption by oscillating bodies

By D. V. EVANS

Department of Mathematics, University of Bristol, England

(Received 19 December 1975)

A theory is given for predicting the absorption of the power in an incident sinusoidal wave train by means of a damped, oscillating, partly or completely submerged body. General expressions for the efficiency of wave absorption when the body oscillates in one or, in some cases, two modes are given. It is shown that 100 % efficiency is possible in some cases. Curves describing the variation of efficiency and amplitude of the body with wavenumber for various bodies are presented.

1. Introduction

In a paper in *Nature* in 1974, Salter described experiments in which he had extracted more than 80 % of the wave power from a two-dimensional sinusoidal wave train using a specially contoured two-dimensional rocking cylinder. The essential features of the Salter cylinder were that it had a circular rear section which did not transmit waves downstream during the motion whilst the front section was contoured so as to reflect as little energy as possible. The efficiency of the device is defined as the proportion of the available power per unit crest length of the incident wave which the cylinder absorbs. This clearly depends on the coupling between the cylinder and the fluid, and will vary with wave frequency.

A complete theory is given here for such devices, based on the usual assumptions of linear water-wave theory and assuming that the body is suspended relative to some stable reference platform by a system of linear springs and dampers which provide restoring forces in addition to any natural buoyancy forces.

In § 3 it is shown how, for a cylinder constrained to oscillate in a single mode, a general expression for the maximum efficiency possible may be obtained without examining the equations of motion of the cylinder. In particular, for cylinders which are symmetrical about the axis of oscillation the maximum efficiency turns out to be 50 %. Equation (4.8) of § 4 shows that knowledge of the solution to the radiation problem, in which the body makes forced oscillations in a given mode, is sufficient to determine a general expression for the efficiency as a function of wavenumber. In §§ 5 and 6 the particular cases of a rolling plate and a heaving or swaying half-immersed circular cylinder are considered using known values for the appropriate added-mass and damping coefficients which are required for computing the efficiency. Section 7 describes the corresponding theory for three-dimensional bodies having a vertical axis of symmetry. The remarkable result that the maximum power that can be absorbed by a heaving body of this type is just $L/2\pi$ times the power per unit crest length in an incident wave of wavelength L is proved. A comparison between the relative efficiencies of a heaving sphere and a heaving half-immersed circular cylinder is made. In § 8 the theory is reworked for two modes of oscillation and it is shown how 100% efficiency is possible in some cases. This is illustrated by curves for the half-immersed and totally immersed circular cylinder oscillating in a combination of heave and sway motions. In the latter case, wide bandwidths occurred at certain values of the wavenumber to which the cylinder was tuned.

2. Formulation

The motion is two-dimensional and Cartesian co-ordinates (x, y) are chosen such that $y = 0$ is the undisturbed free surface with y measured vertically upwards and x to the right. The usual assumptions of the linearized theory of water waves permit the introduction of a velocity potential $\Phi(x, y, t)$ satisfying Laplace's equation in the fluid and the linearized free-surface condition

$$\partial^2\Phi/\partial t^2 + g\,\partial\Phi/\partial y = 0 \quad \text{on} \quad y = 0. \tag{2.1}$$

It is assumed that a small amplitude sinusoidal wave train of frequency ω is incident from $x = +\infty$ upon the body, which is a long cylinder having horizontal generators parallel to the wave crests of the incident wave. The cylinder is situated on or beneath the free surface and is constrained to make small amplitude oscillations in response to the incident wave. The oscillations may be of heave, sway or roll, but not a combination of these. The case of a cylinder oscillating in both heave and sway is considered in § 8. In the absence of waves it is assumed that the cylinder is held in equilibrium by a combination of buoyancy forces and forces due to a spring-and-damper system connected to the cylinder, the latter being capable of extracting energy from the cylinder. In the model chosen by Salter, the cylinder was constrained to make rolling oscillations about a fixed point of itself. The power absorption was measured using an electrical dynamometer consisting of two coils in a magnetic field. Velocity signals from one coil were amplified and sent to the other so as to oppose movement. Velocity and force signals were then multiplied to indicate the power absorbed, which was then compared with wave-height measurements. A possible mechanism for power conversion in full-scale models at sea has also been described by Salter. The rotations of the cylinder will produce unidirectional pulses of water through a special pump described in detail in Salter (1974). In the theoretical treatment described here, it will be assumed that the mechanism for power absorption can be described by a simple linear damper having a resistance to motion which is proportional to velocity. It is unlikely that an actual pump behaves in so simple a fashion and a more realistic model would have to allow for nonlinearities in the pumping mechanism.

On the cylinder we impose the condition that the component of the cylinder velocity normal to itself is equal to the normal velocity of the fluid at that point. Let $\zeta_i(t)$ describe the displacement of the cylinder from its equilibrium position.

Here $i = 1$ and 2 relate to sway and heave motions and $i = 3$ relates to rolling motions. Then ζ_1 and ζ_2 describe the horizontal and vertical displacements respectively of the centre of mass of the cylinder, while ζ_3 describes the angular displacement of the cylinder about its point of rotation. The linearized condition on the equilibrium position of the cylinder is then

$$\partial\Phi/\partial n = \dot{\zeta}_i n_i \tag{2.2}$$

for (x, y) on C, the surface of the cylinder, where $\mathbf{n} = (n_1, n_2)$ is the normal vector from the body into the fluid at the point (x, y) and $n_3 = n_2 x - (y+c) n_1$, where c is the depth of the point of rotation.

It is convenient to eliminate the harmonic time dependence by writing

$$\Phi(x, y, t) = \mathrm{Re}\{\phi(x, y)\, e^{i\omega t}\}. \tag{2.3}$$

We next write the complex-valued time-independent potential $\phi(x, y)$ as

$$\phi(x, y) = \omega^{-1} g A \phi_s + i\omega \xi_i \phi_i, \tag{2.4}$$

where

$$\zeta_i = \mathrm{Re}\{\xi_i e^{i\omega t}\} \tag{2.5}$$

and A is a complex constant. The complex potential ϕ_s is the solution of the scattering problem in which the cylinder is held fixed in an incident wave of unit amplitude potential. The complex potential ϕ_i is the solution to the radiation problem in which a normal velocity Re $\{n_i e^{i\omega t}\}$ is prescribed on the cylinder surface, corresponding to small oscillations of unit amplitude in one of the three modes.

Condition (2.2) is satisfied if

$$\partial\phi_s/\partial n = 0, \quad \partial\phi_i/\partial n = n_i \quad \text{on the cylinder.} \tag{2.6}$$

The wave elevation is given by

$$g^{-1}\partial\Phi(x, 0, t)/\partial t = \mathrm{Re}\{i\omega g^{-1}\phi(x, 0)\, e^{i\omega t}\},$$

so that the incident wave has amplitude $|A|$ if we assume

$$\phi_s \sim (e^{iKx} + Re^{-iKx})\, e^{Ky} \quad \text{as} \quad x \to +\infty, \tag{2.7}$$

where R is the complex reflexion coefficient for the scattering problem. For $x \to -\infty$ we assume

$$\phi_s \sim T e^{iKx+Ky}, \tag{2.8}$$

where T is the complex transmission coefficient for the scattering problem. Here $K = \omega^2/g$.

For $x \to \infty$ we assume

$$\phi_i \sim A_i^+ e^{-iKx+Ky}, \tag{2.9}$$

and for $x \to -\infty$

$$\phi_i \sim A_i^- e^{iKx+Ky}. \tag{2.10}$$

3. The maximum efficiency of power absorption

The efficiency of the system will be defined as the proportion of the available power per unit frontage of the incident wave which is extracted by the body.

This will clearly depend on the details of the coupling between the cylinder and the fluid. Some information about the maximum efficiency possible can be gained, however, without knowing the details of the coupling.

We have, from (2.4), (2.7), (2.9) and (2.10),

$$\phi \sim \begin{cases} (gA/\omega)\,(e^{iKx}+R_1\,e^{-iKx}) & \text{as} \quad x \to +\infty, & (3.1) \\ (gA/\omega)\,T_1\,e^{iKx} & \text{as} \quad x \to -\infty, & (3.2) \end{cases}$$

where $$R_1 = R + iKA_i^+\,\xi_i/A, \quad T_1 = T + iKA_i^-\,\xi_i/A. \tag{3.3}$$

The power per unit length in a sinusoidal two-dimensional progressive wave is the mean energy flux per unit length crossing a vertical plane normal to the direction of the wave, and it can be shown that $R_1\bar{R}_1$ $(T_1\bar{T}_1)$ measures the proportion of power in the reflected (transmitted) wave. It follows that E, the efficiency of the system, is just

$$E = 1 - R_1\,\bar{R}_1 - T_1\,\bar{T}_1. \tag{3.4}$$

(Here a bar over a quantity denotes the complex conjugate.) If the cylinder is held fixed so that $\xi_i = 0$ then from (3.3) and (3.4)

$$E = 1 - R\bar{R} - T\bar{T} = 0, \tag{3.5}$$

showing that wave energy flux is conserved in this case. Recently Newman (1975) has demonstrated a relationship between R, T and A_i which may be written as

$$A_i^+ + \bar{A}_i^+ R + \bar{A}_i^- T = 0 \quad (i = 1, 2, 3). \tag{3.6}$$

If we now use (3.3), (3.5) and (3.6), we can, after some algebra, write (3.4) in the form

$$E = 2\,\mathrm{Re}\,\gamma - |\gamma|^2(1-\delta)^{-1}, \tag{3.7}$$

where $$\gamma = iK\bar{A}_i^+\,\xi_i/A$$

and $$\delta = |A_i^-|^2/(|A_i^+|^2 + |A_i^-|^2). \tag{3.8}$$

The quantity δ, a function of frequency, depends solely on the geometry of the cylinder and cannot be influenced by the particular coupling between fluid and cylinder. The coupling effect occurs in γ through the term ξ_i/A, which must be determined from the equation of motion of the body. If we now maximize the expression (3.7) as a function of γ, we obtain

$$E_{\max} = \gamma_{\max} = 1 - \delta. \tag{3.9}$$

Equation (3.9) is a general result for the maximum efficiency that can be achieved by a given two-dimensional cylinder in a single mode of oscillation. It also follows from (3.3), (3.5) and (3.6) that, when (3.9) is satisfied,

$$|R_1| = \delta, \quad |T_1| = \delta^{\frac{1}{2}}(1-\delta)^{\frac{1}{2}}. \tag{3.10}$$

A highly efficient cylinder is one for which δ is as small as possible. That is, the amplitude of the waves produced at $x = -\infty$ by the forced oscillation of the cylinder in the absence of the incident wave must be as small as possible compared with the amplitude of the waves produced at $x = +\infty$. This is equivalent to the criterion used by Salter in designing an efficient cylinder for which T_1 was as small as possible. The equivalence follows from (3.5) and (3.6), which show that if A_i^- is small then so is T and hence T_1 also. Notice from (3.10) that as $\delta \to 0$ the reflected-wave amplitude tends to zero much faster than the transmitted-wave amplitude. For example, if $E_{\max} = 0{\cdot}9$, then $|R_1| = 0{\cdot}1$ and $|T_1| = 0{\cdot}3$ with $|A_i^+|^2 = 9|A_i^-|^2$. For a body with horizontal symmetry, $A_i^+ = (-1)^i A_i^-$ $(i = 1, 2, 3)$, so that $\delta = \frac{1}{2}$ and the maximum possible efficiency is 50 %. In this case it follows from (3.10) that $|R_1|^2 = |T_1|^2 = \frac{1}{4}$, so that half of the incident wave power is shared equally between the reflected and transmitted waves, the other half being absorbed by the body. This result is consistent with the tests made by Salter (1974) on a vertical vane. He obtained an efficiency of 40 % with 25 % of the incident power being transmitted onwards and 20 % being reflected back.

4. The equation of motion of the cylinder

In formulating the equation of motion of the cylinder we shall assume that the cylinder motion is resisted by mechanical forces which can be modelled by a simple spring-and-damper system. Thus $\zeta_i(t)$ satisfies

$$m\ddot{\zeta}_i = -d\dot{\zeta}_i - k\zeta_i + F_i, \tag{4.1}$$

where d and k are the damper and spring constants. For heave and roll motions k may also include a buoyancy force. The term $d\dot{\zeta}_i$ allows a net amount of work to be done on the cylinder over a period, provided $d \neq 0$. The term F_i is the hydrodynamic 'force' on the cylinder. For $i = 1$ and 2 this force is horizontal and vertical respectively; F_3 is the moment about the point of rotation.

The hydrodynamic forces which do work can conveniently be separated into two parts. Thus we write

$$F_i = F_{is} + F_{ii}, \tag{4.2}$$

where F_{is} is the force acting on the cylinder in the ith direction when it is assumed to be held fixed in the presence of the incident wave and F_{ii} is the force in the ith direction due to the oscillation of the body in that direction in the absence of the incident wave. The latter force can be expressed in the form

$$F_{ii} = -a_{ii}\ddot{\zeta}_i - b_{ii}\dot{\zeta}_i. \tag{4.3}$$

The first term on the right-hand side of (4.3) is that part of the force which is exactly out of phase with the acceleration of the cylinder, so that a_{ii} may be interpreted as an added-mass term describing the increase in inertia of the cylinder due to the fluid. The second term on the right-hand side of (4.3) is that part of the force which is exactly out of phase with the velocity of the cylinder. This term arises because work is done in generating surface waves which radiate away from the cylinder. For motions in an infinite fluid $b_{ii} = 0$.

Now Haskind (1957) has shown that the exciting force F_{is} on the fixed cylinder is directly related to the waves generated by forced oscillation of the cylinder in the ith mode. Thus Newman (1962, equation 37), updating the work of Haskind, has shown that

$$F_{is} = \mathrm{Re}\{\rho g A e^{i\omega t} A_i^+\}. \tag{4.4}$$

Furthermore (Newman 1962, equation 38), the damping coefficients are given in terms of the energy radiated to infinity by the expression

$$b_{ii} = \tfrac{1}{2}\rho\omega(|A_i^+|^2 + |A_i^-|^2) = \tfrac{1}{2}\rho\omega|A_i^+|^2(1-\delta)^{-1}. \tag{4.5}$$

It follows from (2.5) and (4.1)–(4.4) that

$$[k-(m+a_{ii})\,\omega^2 + i(b_{ii}+d)\,\omega]\,\xi_i = \rho g A A_i^+, \tag{4.6}$$

and this equation determines the response of the cylinder to the incident wave.

Now the power per unit length absorbed by the cylinder is the mean rate at which work is being done on the cylinder by the fluid, per unit length. This is

$$\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \dot{\xi}_i F_i\,dt = \tfrac{1}{2}\omega^2 d|\xi_i|^2, \tag{4.7}$$

the only contribution coming from the term $d\dot{\xi}_i$ in (4.1). The power per unit frontage of the incident wave is obtained by computing the mean energy flux per unit length across a vertical plane normal to the wave direction. This is just $\frac{1}{4}\rho g^2|A|^2/\omega$, so that E, the proportion of power absorbed, is

$$\begin{aligned} E &= \frac{2\omega^3}{\rho g^2}\,d\left|\frac{\xi_i}{A}\right|^2 \\ &= \frac{2\omega^3 d\rho|A_i^+|^2}{\{k-(m+a_{ii})\,\omega^2\}^2 + \omega^2(b_{ii}+d)^2} \quad \text{from (4.6)} \\ &= \frac{4\omega^2 d b_{ii}(1-\delta)}{\{k-(m+a_{ii})\,\omega^2\}^2 + \omega^2(b_{ii}+d)^2} \quad \text{from (4.5).} \end{aligned} \tag{4.8}$$

The same result can be obtained, after some algebra, by substituting for ξ_i/A from (4.6) into the alternative expression (3.7) for E, and using (4.5).

We see immediately from (4.8) that for a given cylinder the maximum efficiency occurs when $k = (m+a_{ii})\,\omega^2$ and $d = b_{ii}$, giving $E_{\max} = 1-\delta$ in agreement with (3.9). Equation (4.8) can be used to compute the efficiency for different frequencies of the incident wave once the frequency-dependent terms a_{ii}, b_{ii} and δ are known for the particular cylinder being considered. Notice that for $d = 0$, a freely oscillating cylinder, and $d = \infty$, a fixed cylinder, $E = 0$ as expected. If we assume that the parameters k and d can be varied, then for maximum efficiency at a frequency ω_0, say, we choose $k = [m + a_{ii}(\omega_0)]\,\omega_0$ and $d = b_{ii}(\omega_0)$. Computed values of the added-mass and damping terms a_{ii} and b_{ii} are typically non-dimensionalized by writing

$$a_{ii} = M\mu_i, \quad b_{ii} = M\omega\lambda_i, \tag{4.9}$$

where M is the mass (per unit length) of fluid displaced by a half-immersed circular cylinder having radius a equal to a typical length of the body. If the body is completely submerged, then M is taken to be the mass (per unit length) of fluid displaced by a completely submerged circular cylinder. Thus $M = \frac{1}{2}\pi\rho a^2$ or $\pi\rho a^2$. For roll motions the corresponding moments of inertia are taken for M. We introduce the dimensionless wavenumber $\nu = \omega^2 a/g = 2\pi a/L$, where L is the wavelength of the incident wave and also $\nu_0 = \omega_0^2 a/g$. Then

$$E = \frac{4\nu(1-\delta)\,(\nu^{\frac{1}{2}}\lambda_i)\,(\nu_0^{\frac{1}{2}}\lambda_{i0})}{\{(m'+\mu_{i0})\,\nu_0 - (m'+\mu_i)\,\nu\}^2 + \nu(\nu^{\frac{1}{2}}\lambda_i + \nu_0^{\frac{1}{2}}\lambda_{i0})^2}, \tag{4.10}$$

where

$$\lambda_{i0} = \lambda_i(\nu_0), \quad \mu_{i0} = \mu_i(\nu_0), \quad m' = m/M.$$

In a similar manner, from (4.6) the expression for the ratio of the amplitude of oscillation of the cylinder to the amplitude of the incident wave is given by

$$|\xi_i/A|^2 = 2\rho a^2 M^{-1}(1-\delta)\,\lambda_i/\{[(m'+\mu_{i0})\,\nu_0-(m'+\mu_i)\,\nu]^2+\nu(\nu_0^{\frac{1}{2}}\lambda_{i0}+\nu^{\frac{1}{2}}\lambda_i)^2\}. \tag{4.11}$$

For a symmetric body in heave or sway the numerator in (4.11) is just $2\lambda_i/\pi$ for partly submerged bodies and λ_i/π for completely submerged bodies.

A desirable property of any wave-absorbing device is the ability to operate at high efficiency over a wide bandwidth. In order to test the efficiency for varying ν we need to know the variation of μ_i, λ_i and δ with wavenumber for the particular body. Unfortunately, to the author's knowledge, no computations exist for these parameters for asymmetric two-dimensional cylinders of the type considered by Salter. There do however exist computations for various two-dimensional cylinders having horizontal symmetry as these coefficients are useful in ship hydrodynamics for determining ship motion using strip theory. Before considering particular cylinders in detail, it is necessary to consider more closely the conditions for maximum efficiency. From (4.8) it is seen that this requires

$$d = b_{ii}(\omega), \quad k = [m+a_{ii}(\omega)]\,\omega^2. \qquad (4.12), (4.13)$$

It is assumed that the damping constant d which models the power-absorbing mechanism can be varied such that (4.12) can be satisfied at a given frequency ω_0, say. It remains to check whether (4.13) can be satisfied for $\omega = \omega_0$.

For partially immersed cylinders in roll this can be achieved by varying the distance of the centre of mass of the cylinder below the metacentre since for cylinders with horizontal symmetry about the roll axis, at least, k varies as this distance. Generally, for cylinders in roll, the values of k and d can be adjusted to satisfy (4.12) and (4.13) with $i = 3$, so that the cylinder is 'tuned' to any given non-dimensional wavenumber ν_0. For cylinders in sway, there is no change in the buoyancy forces, so that horizontal springs must be used to provide the restoring force, the stiffness of the springs being chosen to satisfy (4.13) with $i = 1$. The situation for partially immersed cylinders in heave is not so straightforward since, because of the large buoyancy forces which occur, it is not always possible to satisfy (4.13) over the complete frequency range of interest. Thus in the absence of vertical springs, $k = 2b\rho g$ for small vertical oscillations, where $2b$ is the water-line width of the cylinder in equilibrium. Then, in dimensionless form, (4.13) becomes

$$4\pi^{-1}b/a = [m'+\mu_2(\nu)]\,\nu, \tag{4.14}$$

where $m' = m/M$. In general, the variation of the heave added-mass coefficient μ_2 with ν is such that (4.14) has one root ν_1, say, and including vertical springs increases k and hence increases the left-hand side of (4.14). This has the effect of increasing ν_1, so that by varying the stiffness of the springs it is only possible to tune the system to wavenumbers $\nu_0 \geqslant \nu_1$. This lower bound on ν_0 may mean that in order to tune the cylinder to the predominant wavelengths an unacceptably large cylinder would be required (since $\nu = 2\pi a/L$). One way round this is to increase the effective mass of the cylinder, thereby reducing ν_1 in (4.14). A method of doing this without affecting the equilibrium position of the floating cylinder is described by Budal & Falnes (1975). Alternatively the cylinder in heave can be partially tuned by satisfying (4.12) at any desired frequency and allowing k to have its natural value of $2b\rho g$, so that in general (4.13) is not satisfied.

In the next sections we consider the particular cases of a rolling vertical plate and a half-immersed swaying or heaving circular cylinder. In the plate problem we shall be able to tune the plate to any desired wavenumber by varying the position of the centre of mass. This may also be achieved for the swaying circular cylinder by incorporating horizontal springs. For the circular-cylinder problem we shall obtain, for different m', a lower bound to the wavenumbers to which the cylinder can be tuned. The maximum efficiency attainable in each case is 50 %, because of the horizontal symmetry about the axis of oscillation.

5. The rolling vertical plate

Consider a thin vertical plate submerged to a depth a which is constrained to roll about a horizontal axis in the free surface. Since the plate is symmetrical about $x = 0$, $\delta = \frac{1}{2}$ in (4.10) and (4.11). For rolling motions $i = 3$ and $M = \frac{1}{4}\pi\rho a^4$, the moment of inertia per unit length of the fluid displaced by a semicircular cylinder of radius a around its axis.

The problem of the fluid motion produced by the rolling motion of a thin vertical plate is one of the few water-wave problems which permit an explicit solution. Thus in the appendix to a paper by Kotik (1963) a reviewer has obtained explicit expressions in terms of Bessel and Struve functions for the dependence on ν of the added-mass coefficient μ_3 and the damping coefficient λ_3. These were derived from the full solution to the problem given by Ursell (1948). Kotik (1963) provides numerical values for λ_3 and μ_3 over a wide range of ν by making use of the Kramers–Kronig relations. These permit μ_3 to be computed from a Cauchy-type integral involving λ_3, the latter being relatively easy to find since it is related via the Haskind relations to the exciting force on the fixed plate.

Assuming the plate to be uniform, we have

$$m' = m/M = 4sh/3\pi a,$$

where s is the specific gravity of the plate and h its (small) thickness. It follows that in general m' will be small enough for the inertia of the plate to be neglected compared with its added inertia. It is assumed that (4.12) and (4.13) can be satisfied for all frequencies of interest, the latter by varying the position of the centre of mass. In figure 1, equation (4.10), with $\delta = \frac{1}{2}$ and $m' = 0$, has been used to plot the power efficiency E against the dimensionless wavenumber ν for different values of ν_0, the wavenumber for which maximum efficiency is desired. It can be seen that for $\nu_0 = 0{\cdot}3$ the bandwidth is narrow but for $\nu_0 = 0{\cdot}5$, E has a second maximum at about $\nu = 1{\cdot}8$, where the efficiency reaches almost 50 %, the maximum possible in this case since the plate is symmetrical. For $\nu_0 = 0{\cdot}5$ then, there is a wide bandwidth with the efficiency remaining above 40 % for

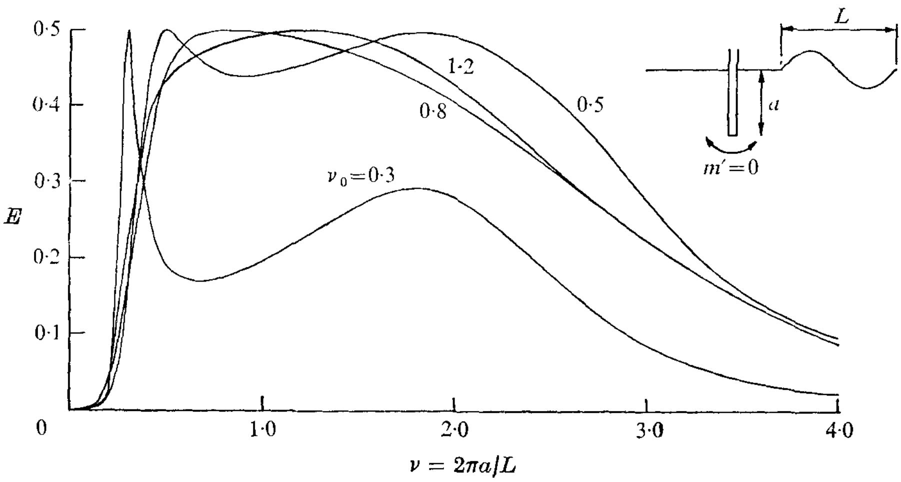

FIGURE 1. Efficiency of power absorption E of a rolling plate with $m = 0$ *vs.* dimensionless wavenumber ν for different values of the tuned wavenumber ν_0.

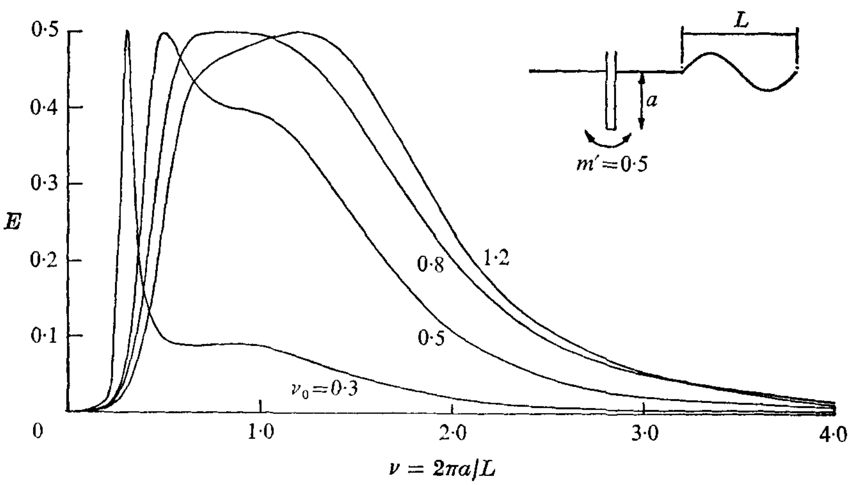

FIGURE 2. Efficiency of power absorption E of a rolling plate with $m = 0·5M$ *vs.* dimensionless wavenumber ν for different values of the tuned wavenumber ν_0.

$\nu = 0·4$–$2·6$. The same is true for $\nu_0 = 0·6$ and to a lesser extent for $\nu_0 = 0·7$ although this is not shown in figure 1. For larger values of ν_0, the second peak gradually disappears, producing narrower bandwidths. It would appear that a vertical plate rolling about an axis in the undisturbed surface operates most efficiently as a wave-power absorber when it is tuned to wavenumbers ν_0 lying between 0·5 and 0·7. For example, a plate of length 20 m tuned for maximum efficiency at $\nu_0 = 0·5$ would be over 40 % efficient in responding to any wave having a wavelength between 50 m and 300 m. On a laboratory scale, a similar efficiency is possible with a plate 15 cm long responding to wavelengths between 40 cm and 160 cm.

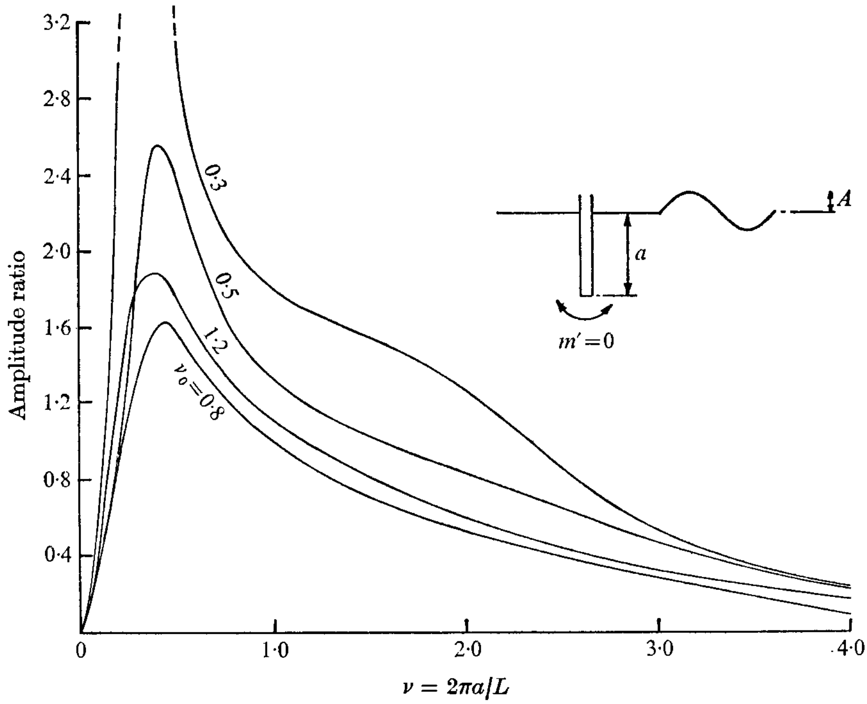

FIGURE 3. Ratio $|\xi_3 a/A|$ of maximum displacement of lower edge of rolling plate with $m = 0$ to incident wave amplitude *vs.* dimensionless wavenumber ν for different values of the tuned wavenumber ν_0.

The effect of increasing the inertia of the plate can be seen in figure 2, where $m' = 0{\cdot}5$. For $\nu_0 = 0{\cdot}5$ the 40 % efficiency bandwidth has been reduced to $\nu = 0{\cdot}42$–$1{\cdot}62$ and the second maximum has disappeared. A better choice of wavenumber to which to tune the system is probably $\nu_0 = 1{\cdot}2$, giving $E > 0{\cdot}4$ for $\nu = 0{\cdot}6$–$1{\cdot}6$. In this case our 20 m plate is over 40 % efficient for waves with wavelengths lying between 80 m and 200 m.

The roll amplitude of the plate as a function of ν is given by (4.11) with $i = 3$, $\delta = \frac{1}{2}$ and $M = \frac{1}{4}\pi\rho a^4$. The complex quantity ξ_3 describes the amplitude and phase of the angular displacement of the plate from the vertical. Figure 3 shows $|\xi_3 a/A|$, the ratio of the maximum displacement of the lower edge of the plate to the incident wave amplitude, plotted against wavenumber ν, for different values of the tuned wavenumber ν_0. For small values of ν_0 large resonant plate amplitudes occur near the tuned wavenumber. For instance for $\nu_0 = 0{\cdot}3$ the maximum displacement of the lower edge of the plate is 7 times the incident wave amplitude for ν close to $0{\cdot}3$. Such large plate displacements are not consistent with a linearized theory and in practice one would expect nonlinear effects such as wave breaking to predominate in this case. For larger values of ν_0, the peak displacement is much smaller and occurs not near the tuned wavenumber but near $\nu = 0{\cdot}4$. As ν increases the displacement diminishes from its peak to zero monotonically. The effect of plate inertia on the plate displacement

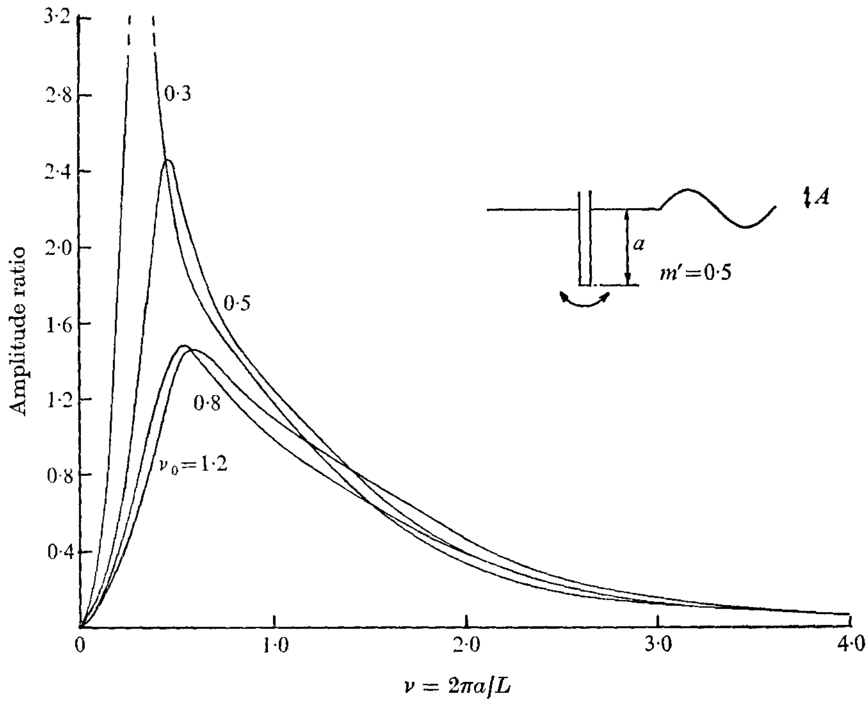

FIGURE 4. Ratio $|\xi_3 a/A|$ of maximum displacement of lower edge of rolling plate with $m = 0{\cdot}5M$ to incident wave amplitude *vs.* dimensionless wavenumber ν for different values of the tuned wavenumber ν_0.

is shown in figure 4, where it can be seen that, for $m/M = 0{\cdot}5$, the plate displacements are diminished and the curves for different values of ν_0 are compressed, there being little difference between them for $\nu > 1{\cdot}4$.

6. The half-immersed swaying or heaving circular cylinder

As a second example we consider a half-immersed circular cylinder which is constrained to make small horizontal (sway) or vertical (heave) oscillations in response to the incident waves. Values of the sway and heave added-mass and damping coefficients were estimated from curves given by Frank (1967), for values of the dimensionless wavenumber ν up to 1·5.

Results for a floating horizontal cylinder constrained to make pure swaying oscillations are shown in figure 5. Since there is no buoyancy restoring force in sway, it is assumed that the cylinder is restrained by horizontal springs whose stiffness k can be varied, along with d, the damping constant, such that both (4.12) and (4.13) can be satisfied for all wavenumbers of interest. Equation (4.10) was computed with $\delta = \frac{1}{2}$, $i = 1$ and $m = M$ for different values of ν and with $\nu_0 = 0{\cdot}3$, $0{\cdot}5$ and $1{\cdot}0$. It can be seen that for $\nu_0 = 0{\cdot}3$ the bandwidth is narrow but that for $\nu_0 = 0{\cdot}5$ and $1{\cdot}0$ it is much wider, so that at $\nu_0 = 0{\cdot}5$, for example, over 40 % of the available power is extracted from waves of wavelength between 5 and 18 times the cylinder radius. When the cylinder is tuned for maximum

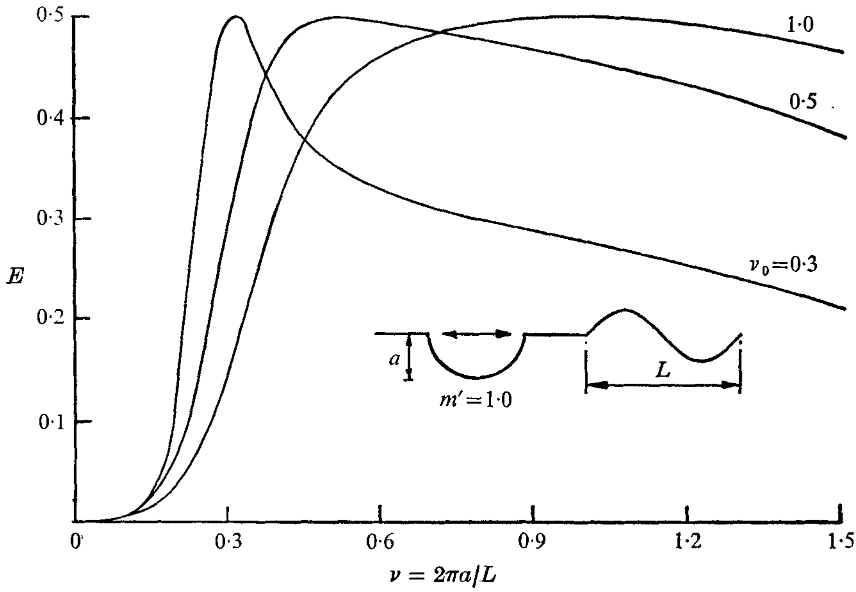

FIGURE 5. Efficiency of power absorption E of a swaying half-immersed circular cylinder with $m = M$ *vs.* dimensionless wavenumber ν for different values of the tuned wavenumber ν_0.

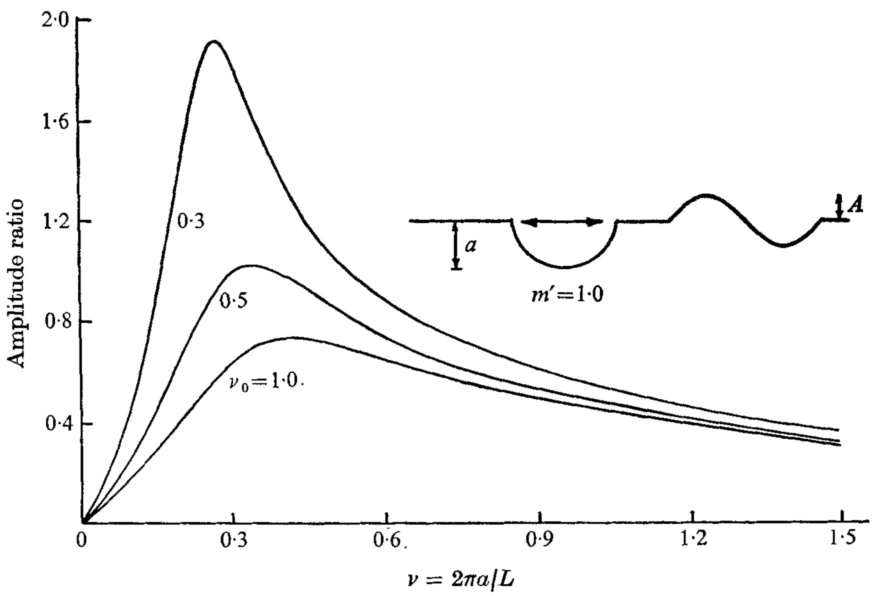

FIGURE 6. Ratio $|\xi_1/A|$ of the sway amplitude of a half-immersed circular cylinder to the incident wave amplitude *vs.* dimensionless wavenumber ν for different values of the tuned wavenumber ν_0.

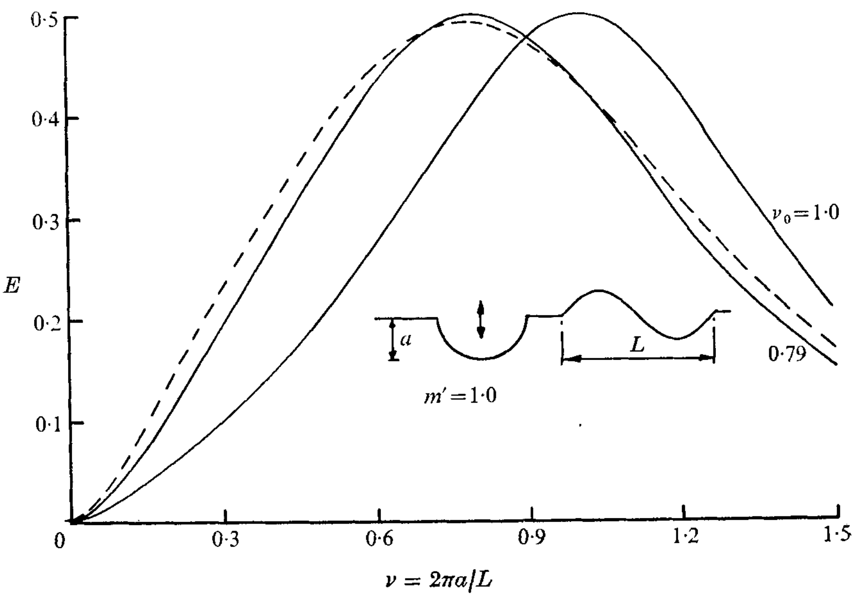

FIGURE 7. Efficiency of power absorption E of a heaving half-immersed circular cylinder with $m = M$ *vs.* dimensionless wavenumber ν for different values of the tuned wavenumber ν_0. The natural tuned wavenumber is $\nu_0 \doteqdot 0{\cdot}79$. The dashed curve shows the effect of partial tuning at $\nu_0 = 0{\cdot}3$.

efficiency at $\nu_0 = 1{\cdot}0$ the efficiency is largely insensitive to small variations in wavenumber about $\nu = 1{\cdot}0$. The corresponding sway amplitude of the cylinder relative to the incident wave amplitude is shown in figure 6. For both long and short waves the sway amplitude tends to zero, so that there is a maximum amplitude for each value of ν_0, the maximum decreasing with increasing ν_0.

For a floating half-immersed circular cylinder making small vertical oscillations there is a natural buoyancy restoring force and $k = 2a\rho g$, so that with $m = M$ equation (4.14) has the root $\nu_1 \doteqdot 0{\cdot}79$. At this wavenumber, corresponding to waves of wavelength about 8 times its radius, the heaving cylinder can be tuned to extract 50 % of the wave power by choosing d to satisfy (4.12). Figure 7 shows how the efficiency of power extraction computed from (4.10) with $\delta = \frac{1}{2}$ and $i = 2$ varies with ν. As for the swaying cylinder, E decreases monotonically either side of the tuned wavenumber and the cylinder is over 40 % efficient for waves of wavelength between about 6 and 11 times the radius of the cylinder. Also shown is the effect of tuning the cylinder to a wavenumber $\nu_0 = 1{\cdot}0$. This is done by choosing k and d such that (4.12) and (4.13) are satisfied by $\nu_0 = 1{\cdot}0$, a vertical spring being required to provide the additional restoring force. The two curves are similar, the second curve showing that the cylinder is now 40 % efficient in absorbing waves of wavelength between about 5 and 8 times the cylinder radius.

It has already been mentioned that a possible method of producing peak efficiencies at lower wavenumbers is to tune the cylinder only partially. In other words we allow k its natural value of $2\rho ga$ so that (4.13) is satisfied by

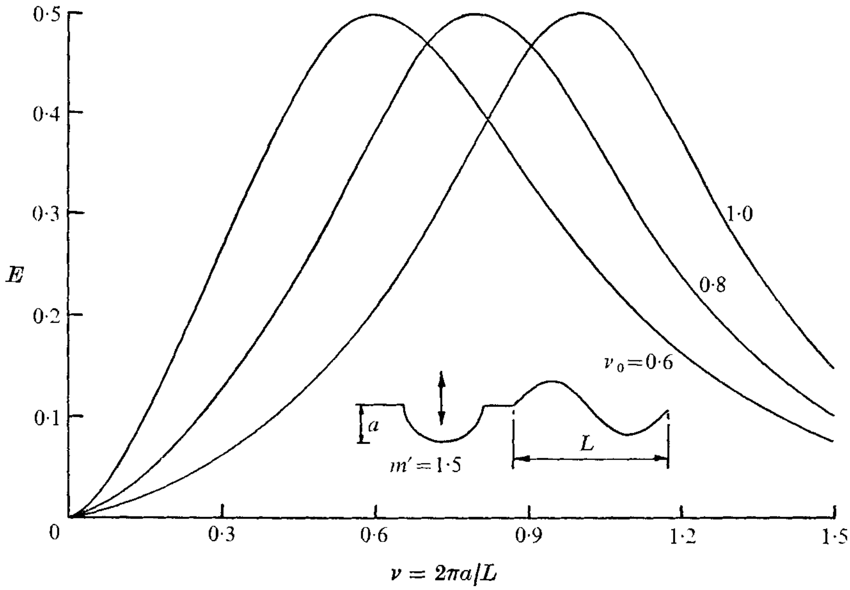

FIGURE 8. Efficiency of power absorption E of a heaving half-immersed circular cylinder with $m = 1{\cdot}5M$ *vs.* dimensionless wavenumber ν for different values of the tuned wavenumber ν_0. The natural tuned wavenumber is $\nu_0 \doteqdot 0{\cdot}6$.

$\nu_1 = 0{\cdot}79$ but adjust d to any desired wavenumber so that (4.12) is satisfied at, say, $\nu = \nu_0$, but not (4.13). In this case the denominator of the expression (4.10) for E must be changed to

$$\{4\pi^{-1} - (m' + \mu_2)\,\nu\}^2 + \nu(\nu^{\frac{1}{2}}\lambda_2 + \nu_0^{\frac{1}{2}}\lambda_{20})^2 \tag{6.1}$$

and a similar change is required in the expression (4.11) for the heave amplitude ratio. The result is shown by the dashed curve in figure 7, where $\nu_0 = 0{\cdot}3$. It is noticeable how little effect this partial tuning has on shifting the peak efficiency away from $\nu = 0{\cdot}79$. The efficiency is increased by about 30 % for $\nu = 0{\cdot}3$ and reduced by only about 2 % for $\nu = 0{\cdot}79$, suggesting that the restoring force is more important than the damping force in determining the peaks of maximum efficiency.

An alternative method of lowering the wavenumber at which maximum efficiency occurs is to increase the effective mass of the cylinder. Thus figure 8 shows the variation of E with ν when $m/M = 1{\cdot}5$. With this value (4.14) has the root $\nu_1 \doteqdot 0{\cdot}6$, so that the cylinder can be tuned to 50 % efficiency in response to waves of wavelength about 10 times the cylinder radius. Also shown in figure 8 is the effect of increasing the 'stiffness' of the cylinder by adding vertical springs so that maximum efficiency is achieved at $\nu_0 = 0{\cdot}8$ and $\nu_0 = 1{\cdot}0$. It can be seen that, just as for the rolling vertical plate, the effect of increasing the inertia is to reduce the efficiency bandwidth. Comparing the corresponding curves for $\nu_0 = 1{\cdot}0$ and $m/M = 1$ and $1{\cdot}5$ respectively shows how the 40 % efficiency bandwidth is reduced from $\nu = 0{\cdot}77$–$1{\cdot}24$ to $\nu = 0{\cdot}82$–$1{\cdot}17$.

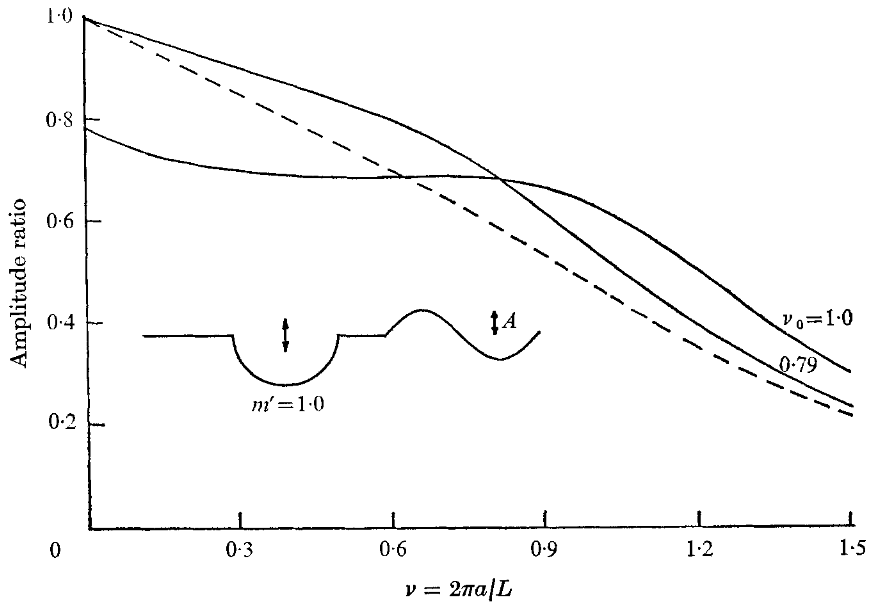

FIGURE 9. Ratio $|\xi_2/A|$ of the heave amplitude of a half-immersed circular cylinder with $m = M$ to the incident wave amplitude *vs.* dimensionless wavenumber ν for different values of the tuned wavenumber ν_0. The dashed curve shows the effect of partial tuning at $\nu_0 = 0{\cdot}3$.

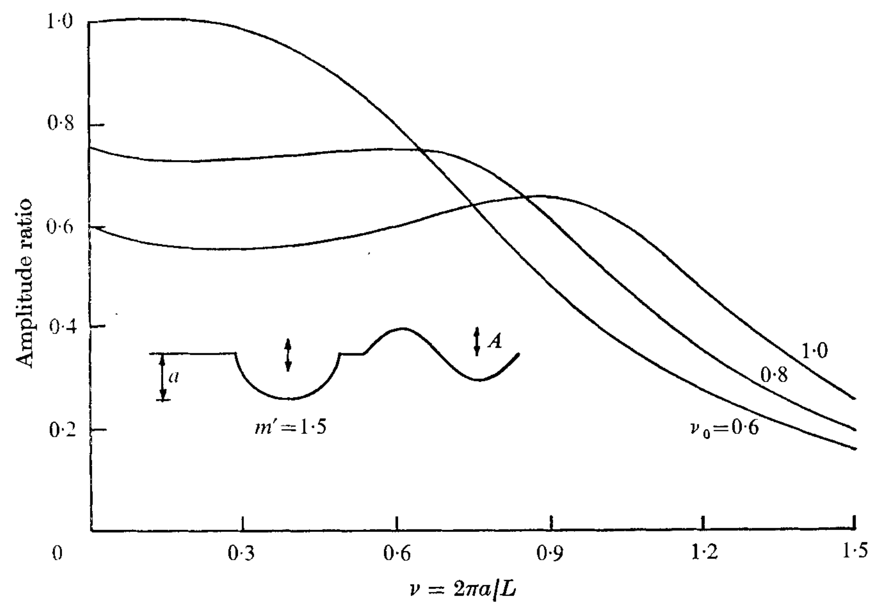

FIGURE 10. Ratio $|\xi_2/A|$ of the heave amplitude of a half-immersed circular cylinder with $m = 1{\cdot}5M$ to the incident wave amplitude *vs.* dimensionless wavenumber ν for different values of the tuned wavenumber ν_0.

The heave amplitude of the cylinder relative to the incident wave amplitude is shown in figure 9. When tuned to the 'natural' wavenumber, $\nu_0 = 0{\cdot}79$, the heave amplitude decreases monotonically with ν. The dashed curve shows the heave amplitude for a partially tuned cylinder with $\nu_0 = 0{\cdot}3$. It is less than the corresponding amplitude for the completely tuned cylinder for all ν. The effect of increasing the stiffness of the cylinder so that $\nu_0 = 1{\cdot}0$ is to decrease the amplitude for $\nu < 0{\cdot}8$ and to increase it for $\nu > 0{\cdot}8$, there now being a local maximum at about $\nu = 0{\cdot}7$. Figure 10 shows the heave-amplitude variation with ν when the effective mass $m = 1{\cdot}5M$. The curves all show a maximum at a wavenumber ν less than the tuned value ν_0. In general the effect of increasing the inertia is to decrease the heave amplitude at a given wavenumber. The values of the amplitude at $\nu = 0$ are obtained from the asymptotic result

$$|\xi_2/A| = 4/\pi(m'+\mu_{20})\,\nu_0, \tag{6.2}$$

which equals unity for a cylinder with no additional restoring force. Equation (6.2) follows from (4.11) when use is made of the result $\lambda_2(0) = 8/\pi$ for the heaving half-immersed cylinder given by Kotik & Mangulis (1962).

7. Three-dimensional wave-power absorbers

The theory given in § 4 can also be applied to three-dimensional bodies having a vertical axis of symmetry. As before it is assumed that the body is constrained to move in a single mode only. The equations of motion are the same as in § 4 up to (4.4), which must be replaced by the relation between the exciting force F_{is} and the damping coefficients for such bodies given by Newman (1962, equations 31–33). Thus

$$F_{is} = \mathrm{Re}\,\{A(2\epsilon\rho g^3\omega^{-3}b_{ii})^{\frac{1}{2}}\,e^{i\omega t}\}, \tag{7.1}$$

where
$$\epsilon = \begin{cases} 2, & i = 1,3, \\ 1, & i = 2. \end{cases}$$

It follows that the power absorbed by the body is

$$\epsilon\omega^{-1}\rho g^3\,db_{ii}\,A^2/\{[k-(m+a_{ii})\,\omega^2]^2+\omega^2(d+b_{ii})^2\}.$$

The total power in an incident wave of unit frontage is $\rho g^2A^2/4\omega$, and the ratio of these quantities provides us with a power absorption length l as defined by Budal & Falnes (1975):

$$\begin{aligned} l &= \frac{4\epsilon g d b_{ii}}{[k-(m+a_{ii})\,\omega^2]^2+\omega^2(d+b_{ii})^2} \\ &= \frac{\epsilon g}{\omega^2}\left\{1-\frac{(k-(m+a_{ii})\,\omega^2)^2+\omega^2(d-b_{ii})^2}{(k-(m+a_{ii})\,\omega^2)^2+\omega^2(d+b_{ii})^2}\right\}. \end{aligned} \tag{7.2}$$

It follows that, for a given wave frequency ω, the maximum value of l is

$$l_{\max} = \epsilon g/\omega^2 = \epsilon L/2\pi, \tag{7.3}$$

obtained by choosing $k = (m+a_{ii})\,\omega^2$ and $d = b_{ii}$ as in the two-dimensional case.

The result (7.2) leads to the remarkable conclusion that a correctly tuned floating body of *any* diameter is capable of absorbing all the power in an incident wave of frontage equal to $\epsilon L/2\pi$, where L is the wavelength of the incident wave. This result has also been quoted by Budal & Falnes (private communication) for the case of heave oscillations. It appears, therefore, that in response to long waves, the tuned body may be more efficient than a two-dimensional cylinder, which must have $l_{\max}$ less than the cylinder length. However, we find that the response amplitude of the tuned body also increases as L increases, so that the assumptions of small oscillations may be violated by the body when tuned to long waves. In fact, from (7.1) the expression corresponding to (4.6) shows that

$$|\xi_i/A|^2 = 2\epsilon\rho g^3 b_{ii}\,\omega^{-3}/\{[k-(m+a_{ii})\,\omega^2]^2+\omega^2(d+b_{ii})^2\}, \tag{7.4}$$

so that at a given wave frequency

$$|\xi_i/A|_{\max} = (\epsilon\rho g^3/2b_{ii}\,\omega^5)^{\frac{1}{2}} = \left(\frac{\epsilon\rho}{2M\lambda_i}\right)^{\frac{1}{2}}\left(\frac{L}{2\pi}\right)^{\frac{3}{2}},$$

where $b_{ii} = M\lambda_i\,\omega$ and it is usual to choose for M the mass of water displaced by a half-immersed sphere of radius a equal to a typical radius of the body. Thus $M = \frac{2}{3}\pi\rho a^3$ (or twice this if the body is completely submerged) and

$$|\xi_i/A|_{\max} = (3\epsilon/4\pi\lambda_i\,\nu^3)^{\frac{1}{2}}. \tag{7.5}$$

We see that this expression may well become large for small values of ν, depending on the precise variation of λ_i with ν.

If we tune the body to a wavenumber $\nu_0 = \omega_0^2 a/g$ by choosing $k = m' + a_{ii}(\omega_0)\,\omega_0^2$ and $d = b_{ii}(\omega_0)$ then the full non-dimensional expression for the amplitude of the body relative to the incident wave amplitude is

$$|\xi_i/A| = \left(\frac{3\epsilon\lambda_i}{\pi\nu}\right)^{\frac{1}{2}}/\{[(m'+\mu_i)\,\nu_0-(m'+\mu_i)\,\nu]^2+\nu(\nu_0^{\frac{1}{2}}\,\lambda_{i0}+\nu^{\frac{1}{2}}\lambda_i)^2\}^{\frac{1}{2}} \tag{7.6}$$

while the non-dimensional power absorption length $l/2a$ is given by

$$\frac{l}{2a} = \frac{2\epsilon(\nu^{\frac{1}{2}}\lambda_i)\,(\nu_0^{\frac{1}{2}}\,\lambda_{i0})}{[(m'+\mu_{i0})\,\nu_0-(m'+\mu_i)\,\nu]^2+\nu(\nu_0^{\frac{1}{2}}\,\lambda_{i0}+\nu^{\frac{1}{2}}\lambda_i)^2}. \tag{7.7}$$

The heaving sphere

We consider in detail the particular case of a sphere constrained to make small heaving oscillations in response to the incident waves. This has also been considered by Budal & Falnes (1975) but their treatment ignores the effect of the diffracted wave field.

For heaving oscillations $i = 2$ and $\epsilon = 1$ in (7.2) and (7.4), and there is a natural buoyancy restoring force such that $k = \pi\rho a^2$ in (7.4). The non-dimensional form of the optimal tuning condition corresponding to (4.14) is now

$$\tfrac{3}{2} = [m'+\mu_2(\nu)]\,\nu, \quad \text{where} \quad m' = m/M, \tag{7.8}$$

and this has the root $\nu_1 \doteqdot 1{\cdot}045$ for $m' = 1$. Thus optimal tuning is possible only for $\nu = \nu_1$ or, by incorporating additional vertical restoring forces, for $\nu \geqslant \nu_1$. This means that a floating sphere can be an efficient wave-power absorber only for waves of wavelength less than about 6 times the sphere radius. If, as in the

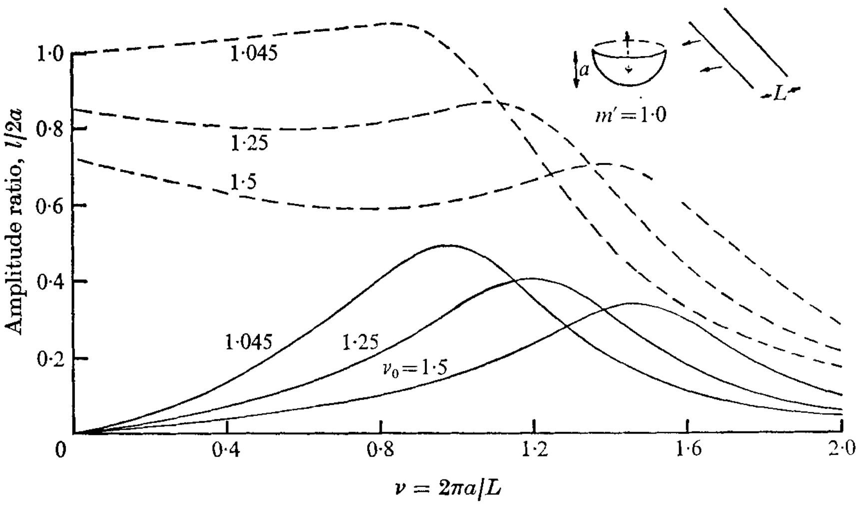

FIGURE 11. Power absorption length ratio $l/2a$ *vs.* dimensionless wavenumber ν for a heaving half-immersed sphere with $m = M$ for different values of the tuned wavenumber ν_0. The natural tuned wavenumber is $\nu_0 \doteqdot 1·045$. The dashed curves are the corresponding ratios $|\xi_2/A|$ of the heave amplitude of the sphere to the incident wave amplitude.

two-dimensional case, we assume that the mass m can be increased without affecting the buoyancy of the sphere, then ν_1 can be reduced. Thus for $m' = 1·5$, equation (7.8) has the root $\nu_1 \doteqdot 0·75$.

Curves of $l/2a$ as a function of ν are shown in figure 11 using values for μ_2 and λ_2 estimated from Havelock (1955). A floating sphere with $m/M = 1·0$ is tuned to wavenumbers $\nu_0 \doteqdot 1·045$. Also shown are the curves obtained when the sphere is tuned to wavenumbers of 1·25 and 1·5 by increasing the vertical restoring force by means of springs. The envelope of the peaks in $l/2a$ has the equation $l/2a = (2\nu)^{-1}$. It can be seen that a heaving sphere can extract all the power in an incident wave whose crest length is equal to just less than half the diameter of the sphere and whose wavelength is about 6 times the radius of the sphere. It is of interest to compare this with the heaving circular cylinder (figure 7), which can extract half of the power in an incident wave of crest length equal to the cylinder length and of wavelength about 8 times the cylinder radius. It follows that, for a circular cylinder to be as efficient as a sphere of diameter $2a$ in extracting power from a given incident wave, it must have length $2a$ and radius $\frac{3}{4}a$ approximately, with $6a = L$, the wavelength of the incident wave. In this case the sphere's mass must be about 1·2 times the mass of the cylinder.

If the sphere is stiffened by increasing the vertical restoring force, the tuned wavenumber increases and the peaks of absorption-length ratio decrease. The heave amplitude ratio is also shown in figure 11 by the dashed curves. The curves peak at a value of ν which is less than the corresponding tuned wavenumber ν_0. For long waves, as $\nu \to 0$, the asymptotic value of $|\xi_2/A|$ is given by

$$|\xi_2/A| \sim 3/[2(m' + \mu_{20})\,\nu_0],$$

which is derived from (7.6) by using the result $\lambda_2(\nu) \sim \frac{3}{4}\pi\nu$ as $\nu \to 0$ given in Kotik & Mangulis (1962) for the heaving sphere. For an unstiffened heaving

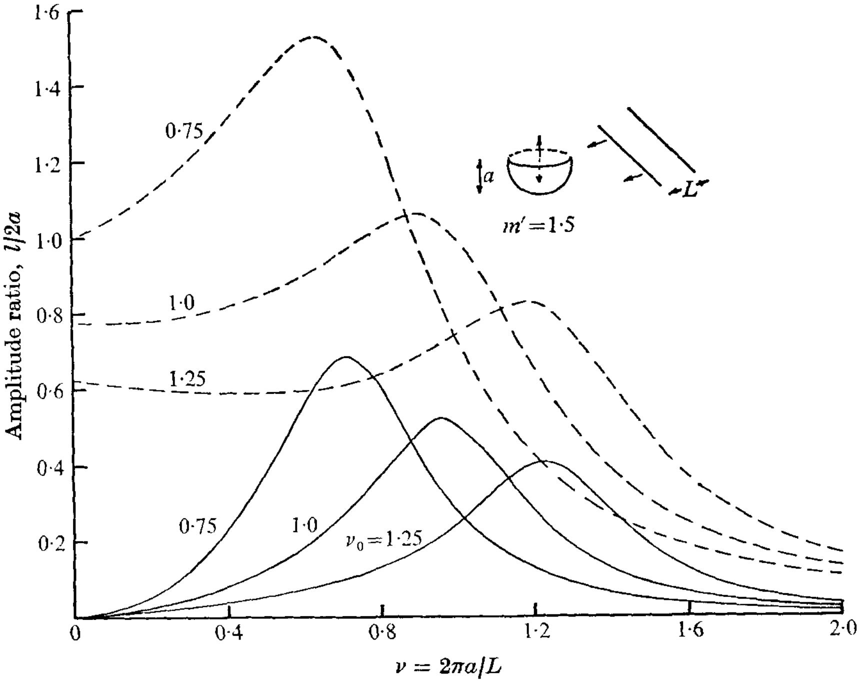

FIGURE 12. Power absorption length ratio $l/2a$ *vs.* dimensionless wavenumber ν for a heaving half-immersed sphere with $m = 1\cdot 5M$ for different values of the tuned wavenumber ν_0. The natural tuned wavenumber is $\nu_0 \doteqdot 0\cdot 75$. The dashed curves are the corresponding ratios $|\xi_2/A|$ of the heave amplitude of the sphere to the incident wave amplitude.

sphere $|\xi_2/A| \sim 1$ as $\nu \to 0$. The effect of increasing the restoring force is to decrease the heave amplitude at a given wavenumber.

Figure 12 shows the effect on $l/2a$ of increasing the mass of the sphere so that $m' = 1\cdot 5$. The natural tuned wavenumber is now $\nu_0 = 0\cdot 75$ and the peak of $l/2a$ is increased accordingly although the bandwidth is narrower. Also the heave-amplitude peak is much larger, being over $1\frac{1}{2}$ times the incident wave amplitude. The curves for $\nu_0 = 1\cdot 0$ and $1\cdot 25$ show that the same is true for the stiffened sphere. The advantage of increasing the mass of the sphere is that it enables tuning to take place at smaller wavenumbers, which means smaller spheres, and also greater power absorption. This is offset, however, by larger heave amplitudes of the sphere, and narrower bandwidths.

The effect of partially tuning the sphere by satisfying $d = b_{22}(\omega)$ for any desired $\omega = \omega_0$ while allowing k to have its natural value of $\pi\rho a^2$ turns out to make little difference to $l/2a$ and is not shown in the figures. As for the circular cylinder, it appears that the restoring force is more important than the damping force in tuning the sphere for maximum power absorption.

8. Oscillations in more than one mode

It was shown in § 3 how the efficiency of power extraction at a given wave-number can be improved by choosing a cylinder for which A_i^- is as small as possible. For a cylinder oscillating about an axis of horizontal symmetry it was shown that the maximum efficiency possible at a given wavenumber was 50 %. In this section we consider cylinders with horizontal symmetry about their axis of oscillation which are allowed to oscillate in more than one mode. To be specific, we assume that the cylinder is held in equilibrium by horizontal and vertical springs and dampers so that, when excited by an incident wave, the cylinder is constrained to make combinations of small horizontal and vertical oscillations without rotation. (Alternative constraints may also be considered but the present assumptions are chosen for simplicity.) We shall show that, at a given frequency, a correctly tuned cylinder, that is, one in which the spring and damper constants have been chosen appropriately, is 100 % efficient as a wave absorber.

The motivation for what follows arises from a paper by Ogilvie (1963), who considered the effect of waves on a completely submerged circular cylinder. In the course of his work he showed that if the centre of the cylinder described a circle then the waves generated by the cylinder motion travelled away from the cylinder along the free surface, but in one direction only. Recent experiments† give a qualitative verification of this result. This phenomenon, while at first appearing remarkable, can be generalized to arbitrary cylinders with horizontal symmetry. Heave oscillations of such a cylinder produce waves of equal amplitude and phase radiating to either infinity whereas sway oscillations produce waves of equal amplitude but exactly out of phase at either infinity. By a suitable combination of the amplitudes and phases of these vertical and horizontal motions it is possible to cancel the wave at one infinity completely, thus producing radiation in one direction only. Reversing the sign of the time coordinate now shows that there exists a motion of the cylinder which will completely absorb a given incident wave.

Having demonstrated in principle the possibility of a 100 % efficient wave absorber, it remains to determine what conditions must be satisfied by the spring and damper constants to ensure that the cylinder will respond in the required fashion and absorb all of the incident wave.

Equations of motion

The equations of motion for the cylinder are very similar to those derived in § 4. Here we allow ζ_1 and ζ_2, the amplitudes of sway and heave respectively, to occur simultaneously. Thus (4.1) is modified to the two equations

$$m\ddot{\zeta}_i = -d_i\dot{\zeta}_i - k_i\zeta_i + F_i \quad (i = 1, 2), \tag{8.1}$$

where d_i and k_i are the damper and spring constants in the horizontal ($i = 1$) and vertical ($i = 2$) directions. As before F_i is the total hydrodynamic force in the ith direction. We may write

$$F_i = F_{is} + \sum_{j=1}^{2} F_{ij} \quad (i = 1, 2),$$

where F_{is} is the force in the ith direction on the cylinder when it is assumed to be fixed in the presence of the incident wave and F_{ij} is the force in the ith direction

† By I. Glendenning (Marchwood Engineering Laboratories, C.E.G.B., Marchwood, Southampton, England), private communication.

due to oscillation of the cylinder in the jth direction. Now for cylinders with horizontal symmetry about their axis of oscillation, $F_{ij} = 0$ if $i \neq j$ $(i, j = 1, 2)$. Because of this the subsequent development follows closely that leading to (4.6). The power per unit length absorbed by the cylinder is modified to include the work done by both F_1 and F_2, to give

$$\frac{\omega}{2\pi}\sum_{i=1}^{2}\int_0^{2\pi/\omega}\dot{\xi}_i F_i\,dt = \tfrac{1}{2}\omega^2\sum_{i=1}^{2} d_i|\xi_i|^2.$$

The expression for the efficiency, after putting $\delta = \frac{1}{2}$ because of the symmetry of the cylinder, becomes

$$E = \sum_{i=1}^{2}\frac{2\omega^2 b_{ii}}{\{k_i-(m+a_{ii})\,\omega^2\}^2+\omega^2(b_{ii}+d_i)^2}. \tag{8.2}$$

An alternative derivation of (8.2) is possible by making use of the definition of E in terms of reflexion and transmission coefficients given by (3.4). In this case the time-independent potential $\phi(x, y)$ can be written as

$$\phi(x,y) = \frac{gA}{\omega}\phi_s + i\omega\sum_{i=1}^{2}\xi_i\phi_i,$$

so that $$R_1 = R+\frac{i\omega^2}{gA}\sum_{i=1}^{2}A_i^+\xi_i,\quad T_1 = T+\frac{i\omega^2}{gA}\sum_{i=1}^{2}A_i^-\xi_i. \qquad (8.3), (8.4)$$

Equations (3.4) and (3.6) now show that

$$E = 2\sum_{i=1}^{2}\{\mathrm{Re}\,\gamma_i - |\gamma_i|^2\}, \tag{8.5}$$

where $\gamma_i = i\omega^2\bar{A}_i^+\xi_i/gA$ and the fact that $|A_i^+| = |A_i^-|$ $(i = 1, 2)$ has been used. The maximum value of E occurs when $\gamma_i = \frac{1}{2}$ $(i = 1, 2)$, whence $E_{\max} = 1$ and the wave is completely absorbed Substituting $\gamma_i = \frac{1}{2}$ into (8.3) and (8.4) we obtain

$$R_1 = R+\tfrac{1}{2}\sum_{i=1}^{2}A_i^+/\bar{A}_i^+,\quad T_1 = T+\tfrac{1}{2}\sum_{i=1}^{2}A_i^-/\bar{A}_i^+.$$

It follows that $R_1 = T_1 = 0$ as expected since $R+(-1)^i T = -(A_i^-/\bar{A}_i^-)$ from (3.6) and $A_i^+ = (-1)^i A_i^-$ $(i = 1, 2)$.

Returning to (8.2) we see that maximum efficiency is achieved at a frequency ω_0 by choosing

$$k_i = (m+a_{ii})\,\omega_0^2,\quad d_i = b_{ii}(\omega_0)\quad (i = 1, 2), \tag{8.6}$$

giving $E = 1$. If we non-dimensionalize the added-mass and damping coefficients in heave and sway by writing $a_{ii} = M\mu_i$ and $b_{ii} = M\omega\lambda_i$ as before, we obtain

$$E = 2\nu\sum_{i=1}^{2}\frac{(\nu^{\frac{1}{2}}\lambda_i)(\nu_0^{\frac{1}{2}}\lambda_{i0})}{\{(m'+\mu_{i0})\,\nu_0-(m'+\mu_i)\,\nu\}^2+\nu(\nu^{\frac{1}{2}}\lambda_i+\nu_0^{\frac{1}{2}}\lambda_{i0})^2} \tag{8.7}$$

in the same notation as in (4.10).

Also, the amplitude of oscillation in the ith mode is given by

$$\xi_i/A = \rho g A_i^+/\{k_i-(m+a_{ii})\,\omega^2+i\omega(d_i+b_{ii})\} \tag{8.8}$$

and (8.6) and (4.5) confirm that $\gamma_i = \frac{1}{2}$ for maximum efficiency.

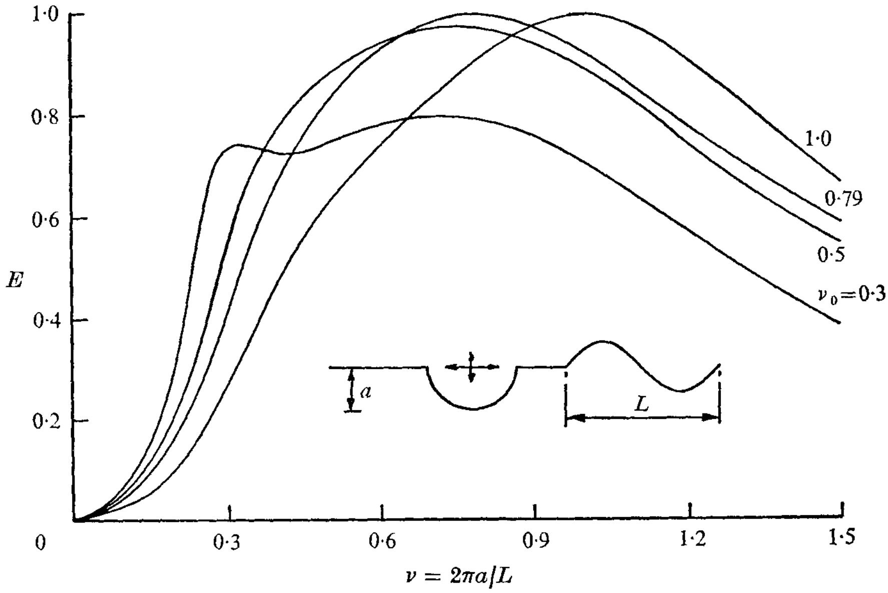

FIGURE 13. Efficiency E of power absorption of a half-immersed circular cylinder with $m = M$ in a combination of heave and sway motions *vs*. dimensionless wavenumber ν for different values of the tuned wavenumber $\nu_0 \geqslant 0{\cdot}79$. For $\nu_0 < 0{\cdot}79$ the effect of partial tuning is shown.

The non-dimensional form of (8.8) is

$$|\xi_i/A|^2 = \frac{\rho a^2}{M}\sum_{i=1}^{2}\lambda_i/[\{(m'+\mu_{i0})\nu_0-(m'+\mu_i)\nu\}^2+\nu(\nu_0^{\frac{1}{2}}\lambda_{i0}+\nu^{\frac{1}{2}}\lambda_i)^2], \tag{8.9}$$

which is similar to (4.11) with $\delta = \frac{1}{2}$.

The half-immersed circular cylinder

We consider combined heave and sway oscillations of a half-immersed circular cylinder. As was pointed out in §4 there is a natural buoyancy force in heave, and in the expression (8.7) for E, the term $(m'+\mu_{20})\nu_0$ must be replaced by $4\pi^{-1}$ for $\nu_0 < \nu_1 \doteqdot 0{\cdot}79$ as in (6.1). For $\nu_0 \geqslant 0{\cdot}79$, (8.7) holds as it stands. Figure 13 shows curves of E, modified as described, against ν for different values of ν_0, the tuned wavenumber. It is assumed that $m = M$ so that the cylinder is floating. For $\nu_0 \geqslant \nu_1$ the horizontal and vertical springs and dampers can be chosen to satisfy (8.6) and 100 % absorption occurs at $\nu = \nu_0$. This can be seen in the curves for $\nu_0 = 0{\cdot}79$ and $\nu_0 = 1{\cdot}0$. For $\nu_0 < \nu_1$ only the horizontal part of the motion can be tuned exactly and so the maximum efficiency attainable is somewhat less than 100 %. At $\nu_0 = 0{\cdot}3$ it can be seen that the maximum efficiency is about 80 % near $\nu = 0{\cdot}7$ and a second, smaller peak occurs near $\nu = 0{\cdot}3$.

The submerged circular cylinder

We next consider the problem of a completely submerged circular cylinder held in equilibrium by horizontal and vertical springs and dampers. For this problem

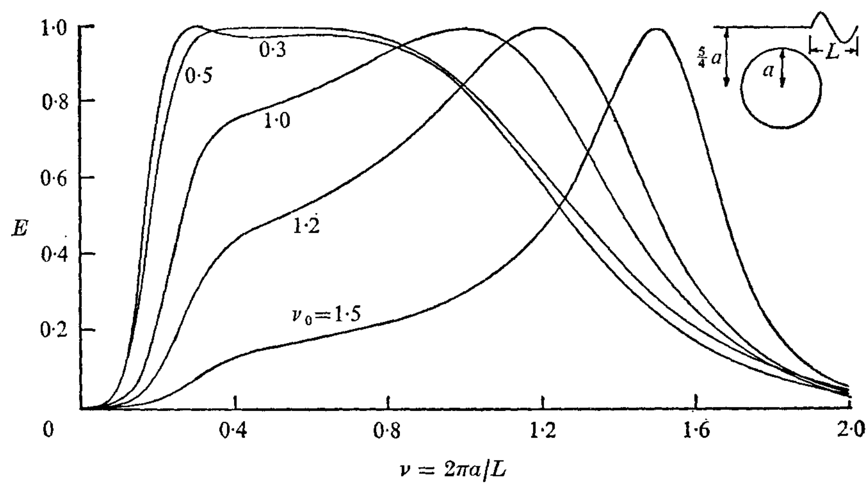

FIGURE 14. Efficiency E of power absorption of a circular cylinder of radius a with $m = M$ submerged to a depth $\frac{5}{4}a$ and making a combination of heave and sway motions *vs.* dimensionless wavenumber ν for different values of the tuned wavenumber ν_0.

the sway and heave added-mass and damping coefficients are the same (Ogilvie 1963, equation 35) and values for these coefficients estimated from the results of Frank (1967) for a cylinder of radius a whose centre is submerged to a depth $\frac{5}{4}a$ are used. For a completely submerged body no natural buoyancy forces occur during the motion so k_i ($i = 1, 2$) can take any desired positive value, so that in general (8.6) can be satisfied. (Ogilvie has shown that a_{ii} can in fact become negative for cylinders close to the surface but this does not occur in the case under consideration.)

Figure 14 shows curves of E against ν for different values of the tuned wavenumber ν_0, computed from (8.7) with $m = M$. At $\nu = \nu_0$ the incident wave is completely absorbed by the cylinder, whose centre is then moving in a circle. For both very short and very long waves, the efficiency of power absorption tends to zero. In the intermediate range a cylinder tuned to a wavenumber $\nu_0 = 0{\cdot}5$–$0{\cdot}7$ is very efficient in extracting power from waves. Thus for $\nu_0 = 0{\cdot}5$ it can be seen that over 90 % of the wave power is extracted from waves of wavelength between about 7 and 24 times the cylinder radius. It is clear that at this value of ν_0 the efficiency of the cylinder is remarkably insensitive to changes in the wavelength of the incident wave.

The heave and sway amplitudes relative to the incident wave amplitude are computed from (8.9) with $M = \pi\rho a^2$, and shown in figure 15 as a function of ν for different ν_0. At $\nu = \nu_0$, the motion of the cylinder is circular but this is not evident from the results as it requires knowledge of the relative phases of the heave and sway motions. For $\nu_0 = 0{\cdot}5$, when the efficiency bandwidth is widest, the heave or sway amplitude has a maximum of about 0·8 times the incident wave amplitude at $\nu = 0{\cdot}25$, corresponding to $E \doteqdot 0{\cdot}87$. At $\nu = 0{\cdot}5$, when $E = 1{\cdot}0$, the amplitude ratio is 0·5, so that the cylinder then describes a circle of radius one-half the incident wave amplitude.

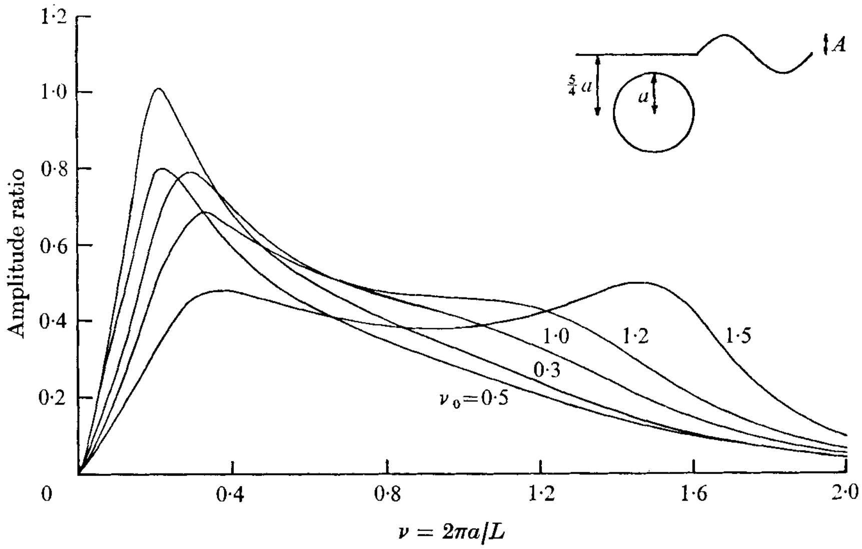

FIGURE 15. Ratio $|\xi_i/A|$ $(i = 1, 2)$ of the heave or sway amplitude to the incident wave amplitude for a circular cylinder of radius a with $m = M$ submerged to a depth $\frac{5}{4}a$ and making a combination of heave and sway motions *vs.* wavenumber ν for different values of the tuned wavenumber ν_0.

The variation of E with depth of submergence of the cylinder can be predicted from (8.7). The damping coefficients for the submerged circular cylinder decay exponentially as the depth of the centre of the cylinder increases relative to the radius of the cylinder. This follows from Ogilvie's (1963) equation (59). Thus one would expect that for deeply submerged cylinders the efficiency would drop sharply either side of $\nu = \nu_0$, producing a very narrow bandwidth. Also, from (8.9) the amplitude ratio at $\nu = \nu_0$ is inversely proportional to the damping coefficient and consequently increases exponentially with decreasing depth.

9. Conclusion

A simple linearized theory has been presented for the absorption of the power in a sinusoidal wave train by an oscillating body. Expressions have been derived for the efficiency of power absorption when the body is a two-dimensional cylinder oscillating in either a single mode or in certain combinations of two modes. These expressions show that the efficiency depends solely on the solution to the radiation problem; namely, when the cylinder is forced to oscillate in the particular mode. A special case of a three-dimensional symmetric body was also considered. Curves were presented showing the variation of efficiency with non-dimensional wavenumber, it being assumed that the body was coupled to the fluid using springs and dampers whose constants could be adjusted. Only simple bodies with symmetry whose wave-making properties were known were used in the calculations. The expressions will be applicable to more efficient asymmetric wave-power absorbers such as the Salter cam once the added-mass and damping coefficients together with the radiation wave amplitudes are known for such bodies.

Of particular interest in the results was the possibility of 100 % efficiency for cylinders oscillating in a combination of modes. The large bandwidth exhibited by the submerged cylinder is encouraging in this respect and may warrant further design studies.

Notes added in proof

(*a*) This paper was presented at the Eleventh Symposium on Naval Hydrodynamics held at University College London in March 1976. A paper presented at the Symposium by Professor J. N. Newman of M.I.T. contains material which overlaps with material in this paper. The same applies to a recently published paper by Professor C. C. Mei of M.I.T. (*J. Ship Res.* **20** (1976), 63–66). In each case the results were derived independently.

(*b*) Dr M. Katory of the British Ship Research Association, Wallsend, has recently published curves of added inertia and damping coefficients for the rolling motion of thè Salter cylinder in the *Naval Architect*, May 1976).

REFERENCES

BUDAL, K. & FALNES, J. 1975 A resonant point absorber of ocean-wave power. *Nature*, **256**, 478–479.

FRANK, W. 1967 Oscillation of cylinders on or below the free surface of deep fluids. *N.S.R.D.C. Rep.* no. 2375.

HASKIND, M. D. 1957 The exciting forces and wetting of ships in waves. *Izv. Akad. Nauk SSSR, Otd. Tekh. Nauk*, **7**, 65–79. (English trans. *David Taylor Model Basin Trans.* no. 307.)

HAVELOCK, T. H. 1955 Waves due to a floating sphere making periodic heaving oscillations. *Proc. Roy. Soc.* A **231**, 1–7.

KOTIK, J. 1963 Damping and inertia coefficients for a rolling or swaying vertical strip. *J. Ship Res.* **7**, 19–23.

KOTIK, J. & MANGULIS, V. 1962 On the Kramers–Kronig relations for ship motions. *Int. Shipbldg Prog.* **9**, 3–10.

NEWMAN, J. N. 1962 The exciting forces on fixed bodies in waves. *J. Ship Res.* **6**, 10–17.

NEWMAN, J. N. 1975 Interaction of waves with two-dimensional obstacles: a relation between the radiation and scattering problems. *J. Fluid Mech.* **71**, 273–282.

OGILVIE, T. F. 1963 First- and second-order forces on a cylinder submerged under a free surface. *J. Fluid Mech.* **16**, 451–472.

SALTER, S. H. 1974 Wave power. *Nature*, **249**, 720–724.

URSELL, F. 1948 On the waves due to the rolling of a ship. *Quart. J. Mech. Appl. Math.* **1**, 246–252.